Quantum Transformation Groupoids

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Motivation

A REMINDER ABOUT GALOIS THEORY:

Let E be a field and G be a finite subgroup of $\mathrm{Aut}(E)$. Then, $E^G \subset E$ is a finite and Galois (normal and separable) extension such that $[E : E^G] = |G|$ (degree of the extension, $\dim_{E^G} E$).

Galois' Theorem

Let $F \subset E$ be a field extension. If the extension is finite and Galois, there is a finite group $G := \mathrm{Gal}(E/F)$ such that $F = E^G$, $|G| = [E : F]$. Moreover, we have a one-to-one correspondence (*Galois correspondence*) between subgroups of G and intermediate fields of $F \subset E$:

(I) For any subgroup $H < G$, $F \subset E^H \subset E$ (intermediate field of $F \subset E$).

(II) For any intermediate field K of $F \subset E$ (*i.e.* $F \subset K \subset E$), $H = \mathrm{Aut}(E/K) < G$ such that $K = E^H$.

ABOUT VON NEUMANN ALGEBRAS AND C*-ALGEBRAS:

Let $\mathcal{H}$ be a Hilbert space and $B \subset \mathcal{B}(\mathcal{H})$ be a $*$-algebra of operators on $\mathcal{H}$.

(1) if B is unital and weak operator closed, B is called a *von Neumann algebra*.

(2) if B is norm operator closed, B is called a *C*-algebra*.

Remarks:

- Any von Neumann algebra is a C*-algebra.
- If B is finite dimensional C*-algebra, then B is also a von Neumann algebra.
- Given $S \subset \mathcal{B}(\mathcal{H})$, we set $S' := \{T \in \mathcal{B}(\mathcal{H}) : T \circ s = s \circ T \text{ for all } s \in S\}$. (Bicommutant theorem) B is a von neumann algebra iff $B'' = B$.

Examples:

(I) Let X be a compact space, then $L^\infty(X)$ is a von Neumann algebra and $C(X)$ is a C*-algebra.

(II) Let $(n_i)_{i=1}^k$ be a finite family of natural numbers, then $B = \bigoplus_{i=1}^k \mathrm{Mat}_{n_i}(\mathbb{C})$ is a finite dimensional C*-algebra. Moreover, any finite dimensional C*-algebra is of this form.

II_1 **SUBFACTORS:**

- Let M be a von Neuman algebra. We say that M is of type II_1 if M is not a finite dimensional algebra and there is a unique (normalize) faithful trace $\tau : M \to \mathbb{C}$ ($\tau(1) = 1$, $\tau(a^* a) = 0 \Rightarrow a = 0, \tau(ab) = \tau(ba)$).
- If M is of type II_1 , then we can see $M \subset \mathcal{B}(L^2_\tau(M))$ where $L^2_\tau(M)$ is the Hilbert space constructed using the scalar product $\langle a|b\rangle := \tau(b^* a)$.
- A II_1 von Neumman algebra M is called a factor if $M' \cap M = \mathbb{C}1$. Here, we calculate the commutator using the inclution $M \subset \mathcal{B}(L^2_\tau(M))$.

Examples:

- Let G be a discrete countable group with the infinite conjugacy class property (a ICC group). Consider the left regular representation $\lambda : G \to \mathcal{B}(l^2(G))$ given by $\lambda(g)\delta_h = \delta_{gh}$ for every $g, h \in G$. Here, we use the notation $\delta_g : h \mapsto \delta_{g,h}$. Then

$$\mathcal{L}(G) := \lambda(G)'' \subset \mathcal{B}(l^2(G)) \qquad \text{(the von Neumann group algebra)}$$

is a II_1 factor. The unique faithful trace $\tau : \mathcal{L}(G) \to \mathbb{C}$ is given by

$$\tau(\lambda(g)) = \langle \lambda(g)\delta_e|\delta_e\rangle = \delta_{g,e}$$

for all $g \in G$.

Remark: *A group G has the ICC property if for every $g \in G - \{e\}$, its conjugacy class is infinite. Example: S_∞ and F_2.*

- For any $n \in \mathbb{N}^*$, we embed the matrix algebra $M_n(\mathbb{C})$ into $M_{2n}(\mathbb{C})$ by send a matrix x to the matrix $\begin{pmatrix} x & 0 \\ 0 & x \end{pmatrix}$. Considering $M_2(\mathbb{C}) \subset M_4(\mathbb{C}) \subset \cdots M_{2^n}(\mathbb{C}) \subset M_{2^{n+1}}(\mathbb{C}) \subset \cdots$ then

$$\mathcal{R} := \overline{\left\{ \bigcup_{n \in \mathbb{N}} \mathrm{M}_{2^n}(\mathbb{C}) \right\}}^{\tau} \qquad \text{where } \tau|_{M_{2^n}(\mathbb{C})} = Tr_n$$

is a II_1 factor. This is called the *hyperfinite* II_1 *factor*.

JONES' INDEX AND JONES' BASIC CONSTRUCTION:

We say that $N \subset M$ is a II_1 subfactor inclusion if N and M are II_1 factors and $\tau_M|_N = \tau_N$. If $N' \cap M = \mathbb{C}1$, we say that the inclusion is irreducible.

Given a II_1 subfactors inclusion $N \subset M$. The Hilbert space $L^2_\tau(M)$ is a left N-module and then we can calculate its Murray-von Neumann dimension

$$[M : N] := \dim_N(L^2_\tau(M)) \qquad \text{(Jones' index)}$$

Remark: In general, $\dim_{\mathbb{C}}(N' \cap M) \leq [M : N]$.

Theorem (Jones '83)

For any inclusion of II_1 factors $N \subset M$

$$[M : N] \in J := \left\{4\cos^2\left(\frac{\pi}{n}\right) : n \geq 3\right\} \cup [4, \infty]$$

Moreover, given $\theta \in J$, then there exists a subfactor $\mathcal{R}_\theta \subset \mathcal{R}$ such that $[\mathcal{R} : \mathcal{R}_\theta] = \theta$.

Given a II_1 subfactor inclusion $N \subset M$, there exists a II_1 factor M_2 such that $N \subset M \subset M_2$, and $[M_2 : M] = [M : N]$. Explicitely, $M_2 =< M, e_1 >= JN'J \subset \mathcal{B}(H_{\tau_M})$, where $J : x \mapsto x^*$ is the conjugation operator coming from the theory of Tomita-Takesaki associate to trace τ_M.

Remarks:

- $N \subset M \subset M_2$ is called the *Jones' basic construction associated to* $N \subset M$.
- We can itirate the basic construction to obtain a tower of II_1 factors:
$$N \subset M \subset M_2 \subset M_3 \subset \cdots \qquad (\textit{Jones' tower associated to } N \subset M).$$
- We will say that the inclusion $N \subset M$ is of depth 2, if
$$M \cap N' \subset M_2 \cap N' \subset M_3 \cap N' \qquad (\textit{derived tower})$$
is the basic construction associated to $M \cap N' \subset M_2 \cap N'$.

Examples:

(I) Let $H \subset G$ be two ICC groups. We have the inclusion of II_1 factors

$$\mathcal{L}(H) \subset \mathcal{L}(G)$$

and $[\mathcal{L}(G) : \mathcal{L}(H)] = [G : H]$.

(II) Let $\alpha : G \to \mathrm{Aut}(M)$ an outer weak continuous action of a finite group G on a II_1 factor M, *i.e.* for each $e \neq g \in G$, we have $\alpha(g) \notin \{\mathrm{ad}(u) = u \bullet u^* : u \text{ is a unitary on } M\}$. Equivalently, $(M^G)' \cap M = \mathbb{C}1$. Then, M^G and $M \rtimes G$ are II_1 factors and

$$[M \rtimes G : M] = [M : M^G] = |G|.$$

(III) Let $\triangleright : H \otimes M \to M$ be a weak continuous action of a finite dimensional Hopf C^*-algebra on M. If the action is outer, *i.e.* $(M^H)' \cap M = \mathbb{C}1$, then M^H and $M \rtimes H$ are II_1 factors and

$$[M \rtimes H : M] = [M : M^H] = [H : \mathbb{C}]$$

Let G be a finite group acting outerly on a II_1 factor M (for example on the hyperfinite II_1 factor $\mathcal{R}$). Then, the inclusion $M^G \subset M$ yields an irreducible depth 2 II_1 subfactor inclusion of finite index $|G|$. Moreover, the *Jones' basic construction of associated to* $M^G \subset M$ is given by the following inclusion of II_1 factors $M^G \subset M \subset M \rtimes G$.

Question: Given an irreducible depth 2 II_1 subfactor inclusion $N \subset M$ of finite index, Is there a group G such that $M^G = N$ and the Jone's basic construction associated to $N \subset M$ is $M^G \subset M \subset M \rtimes G$?

Answer: It was announced by Adrian Ocneanu ('85) that irreducible depth 2 II_1 subfactor inclusions of finite index can be characterized in terms of finite-dimensional Kac algebras (finite quantum groups). This conjecture was achieved with the following theorem:

Ocneanu's theorem (Szymański '94, Longo '94, David '96)

Let $N \subset M$ be an irreducible depth 2 II_1 subfactor inclusion of finite index. Consider its associated Jones' tower $(M_i)_{i\in\mathbb{N}}$ where $M_0 = N$ and $M_1 = M$. Then, there are two finite-dimensional Kac algebra structures on $M' \cap M_3$ and $N' \cap M_2$, dual each other, denoted by $\mathbb{K}$ and $\hat{\mathbb{K}}$ respectively, an outer action of $\mathbb{K}$ on M and an outer action of $\hat{\mathbb{K}}$ on N such that

$$N = M^{\mathbb{K}}, \qquad M_2 \cong M \rtimes \mathbb{K}, \quad \text{and} \quad M \cong N \rtimes \hat{\mathbb{K}}.$$

It seems natural to try to find out if Ocneanu's theorem can be generalize to a more general case, for example if:

- $N \subset M$ is an irreducible depth 2 subfactor inclusion with no restriction on the value of the index;
- $N \subset M$ is a depth 2 II_1 subfactor inclusion of finite index.

Similar results follow in the cases above. And, in order to explain one of the main motivations for the study of operator algebraic quantum groupoids, we give in the following lines a description of these two generalizations of Ocneanu's theorem.

Locally compact quantum groups: Herman & Ocneanu ('98) gave the first steps of a possible generalization for the case of an irreducible depth 2 subfactor inclusion of index not necessarily finite. In their work, they characterize the inclusion of semi-finite factors using crossed product by twisted actions of discrete groups, and they conjecture a result in the case of discrete Kac algebras. The conjecture was finally proved using for it the general framework of operator algebraic quantum groups (mainly the theory of multiplicative unitaries in the sense of Baaj-Skandalis):

Theorem (Enock-Nest '96 + Enock '98)

Let $N \subset M$ be an irreducible depth 2 subfactor inclusion, equipped with a normal semi-finite faithful operator-valued weight T from M to N satisfying some regular condition. Consider its associated Jones' tower $(M_i)_{i\in\mathbb{N}}$ where $M_0 = N$ and $M_1 = M$. Then, there are two locally compact quantum group structures on $M' \cap M_3$ and $N' \cap M_2$, dual to each other, denoted by $\mathbb{G}$ and $\hat{\mathbb{G}}$ respectively; an outer action of $\mathbb{G}$ on M and an outer action of $\hat{\mathbb{G}}$ on N such that

$$N = M^{\mathbb{G}}, \qquad M_2 \cong M \rtimes \mathbb{G}, \quad \text{and} \quad M \cong N \rtimes \hat{\mathbb{G}}.$$

Remark: Any locally compact quantum group arises in that way (Vaes '05).

As corollary, it was shown that:

- If the inclusion $N \subset M$ is compact, the quantum group $\mathbb{G}$ is a compact Kac algebra.
- If the inclusion $N \subset M$ is discrete, the quantum group $\mathbb{G}$ is a discrete Kac algebra.
- If the inclusion $N \subset M$ is compact and discrete (equivalently $N \subset M$ is of finite index), the quantum group $\mathbb{G}$ is a finite-dimensional Kac algebra.

FINITE QUANTUM GROUPOIDS:

It was suggested by Nill, Szlachányi & Wiesbrock ('98) the possibility to characterize finite index depth 2 II_1 subfactor inclusions in terms of finite-dimensional weak Hopf C*-algebras. Weak Hopf C*- algebras and Weak Kac algebras was introduced previously as a generalization of Kac algebras and groupoids algebras.

Similar to Kac algebras, given a finite-dimensional weak Kac algebra $\mathbb{K}$ acting outerly on a II_1 factor M (for example on the hyperfinite II_1 factor $\mathcal{R}$), we obtain a depth 2 II_1 subfactor inclusion $M^{\mathbb{K}} \subset M$ with finite index such that its basic construction is given by

$$M^{\mathbb{K}} \subset M \subset M \rtimes \mathbb{K}.$$

In this case, $\mathbb{K}$ acting outerly on M means that $(M^{\mathbb{K}})' \cap M = C(\mathbb{K})_s$, where $C(\mathbb{K})_s$ denotes the source counit subalgebra of $\mathbb{K}$, then the inclusion above $M^{\mathbb{K}} \subset M$ is not necessarily irreducible since the source counit subalgebra $C(\mathbb{K})_s$ for a weak Kac algebra $\mathbb{K}$ is not necessarily a trivial C*-subalgebra. In fact, $C(\mathbb{K})_s = \mathbb{C}$ if and only if $\mathbb{K}$ is a Kac algebra.

The Ocneanu's theorem has been extended to the framework of finite quantum groupoids (weak Hopf C*-algebras and weak Kac algebras). Moreover, a Galois correspondence was shown for finite depth II_1 subfactor inclusions of finite index. This correspondence makes it possible to share information between finite quantum groupoids and II_1 subfactor inclusions of finite index, for example concerning the categorical data associated with these objects.

Theorem (Nikshych & Vainerman '00 + Nikshych-Vainerman '00)

Let $N \subset M$ be a depth 2 II_1 subfactor inclusion of finite index. Consider its associated Jones' tower $(M_i)_{i\in\mathbb{N}}$ where $M_0 = N$ and $M_1 = M$. Then, there are two finite-dimensional weak Hopf C*-algebra structures on $M' \cap M_3$ and $N' \cap M_2$, dual each other, denoted by $\mathfrak{G}$ and $\hat{\mathfrak{G}}$ respectively, an outer action of $\mathfrak{G}$ on M and an outer action of $\hat{\mathfrak{G}}$ on N such that

$$N = M^{\mathfrak{G}}, \qquad M_2 \cong M \rtimes \mathfrak{G}, \qquad M \cong N \rtimes \hat{\mathfrak{G}},$$

and $[M : N] = \dim(\mathfrak{G}) := \|\Lambda^{C(\mathfrak{G})}_{C(\mathfrak{G})_s}\|^2$. Moreover, we have the equivalences of categories

$${}_N\mathrm{Bim}_N(N \subset M) \cong \mathrm{Rep}(\mathfrak{G}) \quad \text{and} \quad {}_M\mathrm{Bim}_M(M \subset M_2) \cong \mathrm{Rep}(\hat{\mathfrak{G}}).$$

Remark: It can be shown that any finite-dimensional weak Kac algebra arises in that way (Nikshych '98).

MEASURED QUANTUM GROUPOIDS:

A more general question arises from the two generalizations above: Is it possible to give a similar result in the general case of inclusions of von Neumann algebras? Yes.

Theorem (Enock & Vallin '00 + Enock '00 + Enock '05)

Let $N \subset M$ be an inclusion of σ-finite von Neumann algebras of depth 2, equipped with a regular normal semi-finite faithful operator-valued weight T from M to N. Suppose there exists on $N' \cap M$ an adapted faithful semi-finite weight μ and consider the associated Jones' tower $(M_i)_{i \in \mathbb{N}}$ where $M_0 = N$ and $M_1 = M$. Then, there are a measured quantum groupoid structure on $M' \cap M_3$, denoted by $\mathfrak{G} = \mathfrak{G}(N \subset M)$, and an outer action of $\mathfrak{G}$ on M such that

$$N \cong M^{\mathfrak{G}}, \qquad M_2 \cong M \rtimes \mathfrak{G}.$$

Moreover, there are a measured quantum groupoid structure on $N' \cap M_2$, denoted by $\hat{\mathfrak{G}}$, which is the Pontrjagin dual of $\mathfrak{G}$, and an outer action of $\hat{\mathfrak{G}}$ on N such that $M \cong N \rtimes \hat{\mathfrak{G}}$. Using these measured quantum groupoids, the Jones' tower $(M_i)_{i \in \mathbb{N}}$ is given by

$$M^{\mathfrak{G}} \subset M \subset M \rtimes \mathfrak{G} \subset (M \rtimes \mathfrak{G}) \rtimes \hat{\mathfrak{G}} \subset \cdots$$

Remark: Any measured quantum groupoid arises in that way (Enock '11).

Measured quantum transformation groupoids

Given an action α of a measured quantum groupoid $\mathfrak{G}$ on a von Neumann algebra N, then $\alpha(N) \subset N \rtimes_\alpha \mathfrak{G}$ is a depth 2 inclusion of von Neumann algebras satisfying the conditions of the theorem above, then there is a new measured quantum groupoid $\mathfrak{G}(\alpha) := \hat{\mathfrak{G}}(\alpha(N) \subset N \rtimes_\alpha \mathfrak{G})$ such that $\mathfrak{G}(\alpha)$ act on $\alpha(N)$ and $\alpha(N) \rtimes \mathfrak{G}(\alpha) \cong N \rtimes_\alpha \mathfrak{G}$.

In case α is an action of a locally compact quantum group $\mathbb{G}$, by a result of Enock & Timmmermann ('16), there exist a braided commutative Yetter–Drinfeld structure on $\tilde{N} = \alpha(N)' \cap (N \rtimes_\alpha \mathbb{G})$ denoted by $(\tilde{N}, \theta, \widehat{\theta})$ such that $\mathfrak{G}(\alpha) \cong \mathfrak{G}(\tilde{N}, \theta, \widehat{\theta})$ is a measured quantum transformation groupoid.

Open questions: Using the connection between inclusions of von Neumann algebras of depth 2 and measured quantum groupoids:

- Similar to the case of compact/discrete Kac algebras. What kind of inclusions can be found related to compact/discrete quantum transformation groupoids?
- Similar to the case of finite quantum groupoids. Is it possible to give a connection between some categorical data associated to inclusions of von Neumann algebras and compact quantum transformation groupoids?
- Is there a Galois correspondence for actions of compact quantum transformation groupoids on von Neumann algebras that generalizes the known results for compact groups and finite quantum groupoids?

Transformation Quantum Groupoids

Reminder about transformation groupoids

Let G be a group acting on a set S. The set $G \ltimes S := G \times S$ endowed with the applications

$$\begin{array}{rccc} \cdot: & (G \ltimes S)^{(2)} \subseteq (G \ltimes S)^2 & \to & G \ltimes S \\ & ((g,s),(h,t)) & \mapsto & (gh,t) \end{array} \qquad \begin{array}{rccc} ^{-1}: & G \ltimes S & \to & G \ltimes S \\ & (g,s) & \mapsto & (g^{-1}, g \cdot s) \end{array}$$

gives a groupoid which is called *the transformation groupoid associated with the action of G on S*. Here

$$(G \ltimes S)^{(2)} := \Big\{((g,s),(h,t)) \in (G \ltimes S)^2 \,:\, s = h \cdot t\Big\} \subseteq (G \ltimes S) \times (G \ltimes S).$$

Considering

$$\begin{array}{rccc} d: & G \ltimes S & \to & G \ltimes S \\ & (g,s) & \mapsto & (g,s)^{-1}(g,s) = (e,s) \end{array} \qquad \begin{array}{rccc} r: & G \ltimes S & \to & G \ltimes S \\ & (g,s) & \mapsto & (g,s)(g,s)^{-1} = (e, g \cdot s) \end{array}$$

we have

$$(G \ltimes S)^{(2)} = \Big\{((g,s),(h,t)) \in (G \ltimes S)^2 \,:\, d(g,s) = r(h,t)\Big\} = (G \ltimes S)_d \times_r (G \ltimes S)$$

$$(G \ltimes S)^{(0)} := d(G \ltimes S) = r(G \ltimes S) = \{e\} \times S \qquad \text{(unit space)}$$

Remark: If $(G \ltimes S)^{(0)} = \{\bullet\}$, then the groupoid $G \ltimes S$ is the group G .

Two quantum constructions in the litterature:

Lu's Hopf algebroids ('98): Quantum version of a finite transformation groupoid.
Main ingredient: braided commutative Yetter-Drinfeld algebra over a Hopf algebra (Radford '90, Yetter '90, Majid '91).
Advantage: Explicit contruction (Hopf algebroid structure).
Problem: What is its quantum "dual" ? Because there are some problems with the "dual" of an infinite dimensional Hopf algebra.

Enock-Timmermann's measured quantum transformation groupoids ('15): Quantum version of a measured transformation groupoid.
Main ingredient: braided commutative Yetter-Drinfeld von Neumann algebra over a locally compat quantum group (Nest & Voigt '10).
Advantage: Closed by a Pontrjagin-like duality.
Problem: Is there a equivalente C^*-version? A direct translation of the construction is not possible due to the Tomita-Takesaki theory.

Intuitive idea of a quantum transformation groupoid

transformation groupoid

group: G

set: S

action: $G \curvearrowright S$

$G \ltimes S$

S

$d : G \ltimes S \to S$

$r : G \ltimes S \to S$

$\cdot : (G \ltimes S)^{(2)} \to G \ltimes S$

$+$

conditions

Measure: $\nu : S \to \mathbb{C}$

Haar system: $\{\lambda^s\}_{s \in S}$

quantum tranformation groupoid

quantum group: $\mathbb{G}$

algebra: B

two actions: $\begin{matrix} \mathbb{G} \overset{\alpha}{\curvearrowright} B \\ \widehat{\mathbb{G}} \overset{\widehat{\alpha}}{\curvearrowright} B \end{matrix}$ + bc Yetter-Drinfeld condition

$\widehat{\mathbb{G}} \ltimes_{\widehat{\alpha}} B$ (total algebra)

B (base algebra)

$\widehat{\alpha} : B \hookrightarrow \widehat{\mathbb{G}} \ltimes_{\widehat{\alpha}} B$

$\beta_\alpha : B^{op} \hookrightarrow \widehat{\mathbb{G}} \ltimes_{\widehat{\alpha}} B$

$\Delta : \widehat{\mathbb{G}} \ltimes B \to (\widehat{\mathbb{G}} \ltimes B) {}_{\widehat{\alpha}} \times {}_{\beta_\alpha} (\widehat{\mathbb{G}} \ltimes B)$

$+$

conditions

base integral: $\mu : B \to \mathbb{C}$

partial integral: $E : \widehat{\mathbb{G}} \ltimes_{\widehat{\alpha}} B \to B$

Definitions and conventions:

- $\mathbb{G}$ is called an algebraic quantum group, in the sense of Van Daele, if $\mathbb{G} = (\mathcal{O}(\mathbb{G}), \Delta, \varphi)$, where $(\mathcal{O}(\mathbb{G}), \Delta)$ is a multiplier Hopf $*$-algebra and $\varphi : \mathcal{O}(\mathbb{G}) \to \mathbb{C}$ is a left invariant integral (positive faithful functional such that $(id \otimes \varphi)\Delta(a) = \varphi(a)1$ for all $a \in \mathcal{O}(\mathbb{G})$). Using φ, we can construct the dual algebraic quantum group $\widehat{\mathbb{G}}$ and the algebraic multiplicative unitary U (this object encodes the canonical pairing).
- A triplet $(N, \theta, \widehat{\theta})$ is called a *Yetter-Drinfeld* $\mathbb{G}$*-$*$-algebra*, if N is a $*$-algebra, $\theta : N \to \mathrm{M}(\mathcal{O}(\mathbb{G}) \otimes N)$ is an action of $\mathbb{G}$ and $\widehat{\theta} : N \to \mathrm{M}(\widehat{\mathcal{O}(\mathbb{G})} \otimes N)$ is an action of $\mathbb{G}$ such that

$$(\mathrm{id}_{\widehat{\mathcal{O}(\mathbb{G})}} \otimes \theta) \circ \widehat{\theta} = (\Sigma \otimes \mathrm{id}_N) \circ (\mathrm{Ad}(U) \otimes \mathrm{id}_N) \circ (\mathrm{id}_{\mathcal{O}(\mathbb{G})} \otimes \widehat{\theta}) \circ \theta. \qquad \text{(YD)}$$

  If moreover for each $m, n \in N$, we have

$$\theta^{\mathrm{c}}(m^{\mathrm{op}})\widehat{\theta^{\circ}}(n^{\mathrm{op}}) = \widehat{\theta}^{\circ}(n^{\mathrm{op}})\theta^{\mathrm{c}}(m^{\mathrm{op}}) \qquad \text{(BC)}$$

  inside $\mathrm{M}(\mathcal{H}(\mathbb{G}) \otimes N^{\mathrm{op}})$, we say that $(N, \theta, \widehat{\theta})$ is *braided commutative Yetter-Drinfeld* $\mathbb{G}$*-$*$-algebra*. Here $\theta^{\mathrm{c}} := ({}^{\mathrm{op}} \otimes {}^{\mathrm{op}}) \circ \theta \circ {}^{\mathrm{op}}$ and $\widehat{\theta^{\circ}} := (S_{\widehat{\mathbb{G}}} \otimes {}^{\mathrm{op}}) \circ \theta \circ {}^{\mathrm{op}}$.
  Remark: That is equivalente to say that $(N, \triangleleft_{\widehat{\theta}}, \theta)$, where $\triangleleft_{\widehat{\theta}} : B \otimes \mathcal{O}(\mathbb{G}) \to B$ is the dual action of $\widehat{\theta}$, is a braided commutative Yetter–Drinfeld $*$-algebra over the multiplier Hopf $*$-algebra $(\mathcal{O}(\mathbb{G}), \Delta)$ (T. '22)

Measured Yetter-Drinfeld algebras

For simplicity, from now on we will suppose that $\mathbb{G}$ is of compact type, *i.e.* $(\mathcal{O}(\mathbb{G}), \Delta)$ is a unital Hopf $*$-algebra, and N will be a unital $*$-algebra.

A Yetter–Drinfeld $\mathbb{G}$-$*$-algebra $(N, \alpha, \widehat{\theta})$ is called measured, if there is a Yetter–Drinfeld integral μ, *i.e.* a non-zero positive faithful functional $\mu : N \to \mathbb{C}$ such that

$$(\mathrm{id} \otimes \mu)\alpha = \mu(-)1 \text{ (θ-invariant)}, \qquad \text{and} \quad (\mathrm{id} \otimes \mu)\widehat{\alpha} = \mu(-)1 \text{ ($\widehat{\theta}$-invariant)}.$$

Theorem (Canonical automorphisms of a Yetter–Drinfeld $*$-algebra. T. '23)

Let $(N, \theta, \widehat{\theta})$ be a unital braided commutative Yetter–Drinfeld $\mathbb{G}^{\mathrm{c}}$-$*$-algebra. The linear maps

$$\gamma_\theta : \begin{array}{ccc} N & \to & N \\ m & \mapsto & m_{[0]} \triangleleft_{\widehat{\theta}} S_{\mathbb{G}}^{-1}(m_{[-1]}) \end{array} \quad \text{and} \quad \hat{\gamma}_\theta : \begin{array}{ccc} N & \to & N \\ m & \mapsto & m_{[0]} \triangleleft_{\widehat{\theta}} S_{\mathbb{G}}^{2}(m_{[-1]}) \end{array}$$

are "canonical" automorphisms satisfying $\gamma_\theta^{-1} = \hat{\gamma}_\theta$, $\gamma_\theta \circ * \circ \gamma_\theta \circ * = \mathrm{id}$,

$$\widehat{\theta} \circ \gamma_\theta = (S^2_{\widehat{\mathbb{G}}^\circ} \otimes \gamma_\theta) \circ \widehat{\theta} \quad \text{and} \quad \widehat{\theta} \circ \hat{\gamma}_\theta = (S^{-2}_{\widehat{\mathbb{G}}^\circ} \otimes \hat{\gamma}_\theta) \circ \widehat{\theta}.$$

Let $(N, \theta, \widehat{\theta})$ be a unital braided commutative Yetter–Drinfeld $\mathbb{G}^{\mathrm{c}}$-$*$-algebra. Using the canonical automorphism $\hat{\gamma}_\theta$ on N, consider the $\hat{\gamma}_\theta$-opposite $*$-algebra $N^{\mathrm{op}}_{\hat{\gamma}_\theta}$, i.e. the vector space N with non-degenerate $*$-algebra structure given by $m^{\mathrm{op}} n^{\mathrm{op}} := (nm)^{\mathrm{op}}$ and $(m^{\mathrm{op}})^* := \hat{\gamma}_\theta(m^*)^{\mathrm{op}}$ for all $m, n \in N$. By the last theorem, we have

$$\theta \circ \hat{\gamma}_\theta = (S^{-2}_{\mathbb{G}^{\mathrm{c}}} \otimes \hat{\gamma}_\theta) \circ \theta \quad \text{and} \quad \hat{\theta} \circ \hat{\gamma}_\theta = (S^{-2}_{\widehat{\mathbb{G}}^{\circ}} \otimes \hat{\gamma}_\theta) \circ \hat{\theta},$$

thus we can contruct conjugate actions

$$\theta^{\mathrm{c}} : N^{\mathrm{op}}_{\hat{\gamma}_\theta} \to \mathcal{O}(\mathbb{G}) \otimes N^{\mathrm{op}}_{\hat{\gamma}_\theta}, \quad m^{\mathrm{op}} \mapsto (^{\mathrm{op}} \otimes {}^{\mathrm{op}})\theta(m)$$

and

$$\widehat{\theta}^{\mathrm{c}} : N^{\mathrm{op}}_{\hat{\gamma}_\theta} \to \mathrm{M}(\widehat{\mathcal{O}(\mathbb{G})}^{\mathrm{op}} \otimes N^{\mathrm{op}}_{\hat{\gamma}_\theta}), \quad m^{\mathrm{op}} \mapsto (^{\mathrm{op}} \otimes {}^{\mathrm{op}})\widehat{\theta}(m).$$

Theorem (Dual Yetter-Drinfeld $*$-algebras. T. '22 + T. '23)

The following statements are equivalent:

(I) $(N, \theta, \widehat{\theta})$ is a unital braided commutative Yetter–Drinfeld $\mathbb{G}^{\mathrm{c}}$-$*$-algebra with Yetter–Drinfeld integral μ.

(II) $(N^{\mathrm{op}}_{\hat{\gamma}_\theta}, \widehat{\theta}^{\mathrm{c}}, \theta^{\mathrm{c}})$ is a unital braided commutative Yetter–Drinfeld $\widehat{\mathbb{G}}^{\mathrm{c},\circ}$-$*$-algebra with Yetter–Drinfeld integral $\mu^\circ := \mu \circ {}^{\mathrm{op}}$.

Algebraic quantum transformation groupoids (AQTG$^{\mathrm{d}}$)

Let $\mathbb{G} = (\mathcal{O}(\mathbb{G}), \Delta_{\mathbb{G}}, \varphi_{\mathbb{G}})$ be an algebraic quantum group of compact type and $(N, \theta, \widehat{\theta}, \mu)$ be a unital braided commutative measured Yetter–Drinfeld $\mathbb{G}^{\mathrm{c}}$-$*$-algebra with canonical automorphisms denoted by γ_θ and $\hat{\gamma}_\theta$. Consider the unital $*$-algebra $A = \mathcal{O}(\mathbb{G}) \#_{\widehat{\theta}} N \cong \widehat{\mathbb{G}}^{\circ} \ltimes_{\widehat{\theta}} N$, the injective linear maps

$$\begin{array}{rccc} \alpha: & N & \to & A \\ & m & \mapsto & 1_{\mathcal{O}(\mathbb{G})} \# m \end{array}, \qquad \begin{array}{rccc} \beta: & N & \to & A \\ & m & \mapsto & m_{[-1]} \# m_{[0]} \end{array}$$

and the following linear maps

$$\begin{array}{rccc} t_B: & B := \alpha(A) & \to & C := \beta(A) \\ & \alpha(m) & \mapsto & \beta(m) \end{array}, \qquad \begin{array}{rccc} t_C: & C & \to & B \\ & \beta(m) & \mapsto & \alpha(\gamma_\theta(m)) \end{array},$$

$$\begin{array}{rccc} \Delta_B: & A & \to & A \,{}_B\overline{\times}^B\, A \\ & h \# m & \mapsto & (h_{(1)} \# 1_N) \,{}_B\overline{\times}^B\, (h_{(2)} \# m) \end{array},$$

$$\begin{array}{rccc} \Delta_C: & A & \to & A \,{}^C\overline{\times}_C\, A \\ & h \# m & \mapsto & (h_{(1)} \# 1_N) \,{}^C\overline{\times}_C\, (h_{(2)} \# m) \end{array},$$

$$\begin{array}{rccc} S: & A & \to & A \\ & h \# m & \mapsto & \beta(\hat{\gamma}_\theta(m))(S_{\mathbb{G}}(h) \# 1_N) \end{array},$$

$$\begin{array}{rccc} \varepsilon_B : & A & \to & B \\ & \alpha(m)(h \# 1_N) & \mapsto & \alpha(m \triangleleft_{\widehat{\theta}} h) \end{array}, \quad \begin{array}{rccc} {}_C\varepsilon : & A & \to & C \\ & (h \# 1_N)\beta(m) & \mapsto & \beta(m \triangleleft_{\widehat{\theta}} S_{\mathbb{G}}(h)) \end{array},$$

$$\begin{array}{rccc} \mu_B : & B & \to & \mathbb{C} \\ & \alpha(m) & \mapsto & \mu(m) \end{array}, \quad \begin{array}{rccc} \mu_C : & C & \to & \mathbb{C} \\ & \beta(m) & \mapsto & \mu(m) \end{array},$$

$$\begin{array}{rccc} {}_B\psi_B : & A & \to & B \\ & \alpha(m)(h \# 1_N) & \mapsto & \varphi_{\mathbb{G}}(h)\alpha(m) \end{array}, \quad \begin{array}{rccc} {}_C\phi_C : & A & \to & C \\ & (h \# 1_N)\beta(m) & \mapsto & \varphi_{\mathbb{G}}(h)\beta(m) \end{array}.$$

Then, we have:

Theorem (AQTG$^{\mathrm{d}}$ of Compact Type, T. '18 + T. '23)

The collection

$$\mathcal{A}(N, \theta, \widehat{\theta}, \mu) := (A, B, C, t_B, t_C, \Delta_B, \Delta_C, \mu_B, \mu_C, {}_B\psi_B, {}_C\phi_C)$$

yields a unital measured Hopf $*$-algebroid, called the *algebraic quantum transformation groupoid of compact type associated with the braided commutative Yetter–Drinfeld $\mathbb{G}^{\mathrm{c}}$-$*$-algebra $(N, \theta, \widehat{\theta})$ and the Yetter–Drinfeld integral μ.*

Consider now the unital braided commutative measured Yetter–Drinfeld $\widehat{\mathbb{G}}^{\mathrm{c},\circ}$-$*$-algebra $(N^{\mathrm{op}}_{\hat{\gamma}_\theta}, \widehat{\theta}^{\mathrm{c}}, \theta^{\mathrm{c}}, \mu^\circ)$. Then:

Theorem (AQTG$^{\mathrm{d}}$ of Discrete Type, T. '18 + T. '23)

There is a measured multiplier Hopf $*$-algebroid $\mathcal{A}(N^{\mathrm{op}}_{\hat{\gamma}_\theta}, \widehat{\theta}^{\mathrm{c}}, \theta^{\mathrm{c}}, \mu^\circ)$ with total algebra given by the non-degenerate $*$-algebra $\widehat{\mathcal{O}(\mathbb{G})} \#_{\theta^{\mathrm{c}}} N^{\mathrm{op}}_{\hat{\gamma}_\theta} \cong \mathbb{G} \ltimes_{\theta^{\mathrm{c}}} N^{\mathrm{op}}_{\hat{\gamma}_\theta}$. Moreover

(1) The linear map

$$\begin{array}{cccc} \mathcal{T} & \widehat{A} & \to & \widehat{\mathcal{O}(\mathbb{G})} \#_{\theta^{\mathrm{c}}} N^{\mathrm{op}}_{\hat{\gamma}_\theta} \\ & (\alpha(m)(h \# 1_N)) \cdot \phi & \mapsto & (h \cdot \varphi_{\mathbb{G}}) \# \hat{\gamma}_\theta(m)^{\mathrm{op}} \end{array}$$

yields an isomorphism between the measured multiplier Hopf $*$-algebroids

$$\widehat{\mathcal{A}}(N, \theta, \widehat{\theta}, \mu) \quad \text{and} \quad \mathcal{A}(N^{\mathrm{op}}_{\hat{\gamma}_\theta}, \widehat{\theta}^{\mathrm{c}}, \theta^{\mathrm{c}}, \mu^\circ)$$

satisfying $\mathcal{T} \circ \widehat{S} = S' \circ \mathcal{T}$.

(2) The bilinear map

$$\begin{array}{cccc} \mathbb{P}_{\widehat{\theta}, \theta^{\mathrm{c}}} : & \mathcal{O}(\mathbb{G}) \#_{\widehat{\theta}} N \times \widehat{\mathcal{O}(\mathbb{G})} \#_{\theta^{\mathrm{c}}} N^{\mathrm{op}}_{\hat{\gamma}_\theta} & \to & \mathbb{C} \\ & (h \# m) \times (\omega \# n^{\mathrm{op}}) & \mapsto & \mathbf{p}(h, \omega)\mu(nm) \end{array}$$

yields a pairing in the sense of Timmermann, Van Daele & Wang ('22) between the measured multiplier Hopf $*$-algebroids

$$\mathcal{A}(N, \theta, \widehat{\theta}, \mu) \quad \text{and} \quad \mathcal{A}(N^{\mathrm{op}}_{\hat{\gamma}_\theta}, \widehat{\theta}^{\mathrm{c}}, \theta^{\mathrm{c}}, \mu^\circ).$$

Examples:

Classic transformation groupoids: Let G be a finite group acting by the left on a finite space X and $\nu : X \to \mathbb{R}_0^+$ be a non-zero G-invariant function. Consider the unital braided commutative measured Yetter–Drinfeld $\mathbb{G}^{\mathrm{c}}$-$*$-algebra $(K(X), \theta, \hat{\theta} = \mathrm{trv}, \mu_\nu)$ arising from the action of G on X. The measured multiplier Hopf $*$-algebroid $\mathcal{A}(K(X), \theta, \widehat{\theta}, \mu_\nu)$ is given by

$$\begin{array}{rcccl} A = & K(G) \otimes_{\widehat{\theta}} K(X) & \cong & K(G \ltimes X) & \\ & p \otimes f & \mapsto & \displaystyle\sum_{g \in G, x \in X} p(g) f(x) \delta_{(g,x)} & , \end{array}$$

$$\begin{array}{rcccl} \alpha_{\widehat{\theta}}: & K(X) & \to & \mathrm{M}(K(G) \otimes_{\widehat{\theta}} K(X)) & \\ & f & \mapsto & \displaystyle\sum_{g \in G, x \in X} f(d(g,x)) \delta_{(g,x)} & , \end{array}$$

$$\begin{array}{rcccl} \beta_{\theta}: & K(X) & \to & \mathrm{M}(K(G) \otimes_{\widehat{\theta}} K(X)) & \\ & f & \mapsto & \displaystyle\sum_{g \in G, x \in X} f(r(g,x)) \delta_{(g,x)} & , \end{array}$$

$$B := \{\alpha_{\widehat{\theta}}(f) = d^\bullet(f) : f \in K(X)\} \cong K((G \ltimes X)^{(0)}) \cong \{\beta_\theta(f) = r^\bullet(f) : f \in K(X)\} =: C,$$

Given $p \otimes f \in K(G \ltimes X)$,

- $A\,{}_B\overline{\times}^B\,A = A\,{}^C\overline{\times}_C\,A = K((G \ltimes X)^{(2)})$, and
$$\Delta_B(p \otimes f)((g,x),(g',x')) = (p \otimes f)((g,x)(g',x'))$$
for all $((g,x),(g',x')) \in (G \ltimes X)^{(2)}$.
- $S(p \otimes f)(g,x) = (p \otimes f)((g,x)^{-1})$ for all $(g,x) \in G \ltimes X$.

The Pontrjagin dual of the measured multiplier Hopf $*$-algebroid $\mathcal{A}(K(X), \theta, \widehat{\theta}, \mu_\nu)$ is the measured multiplier Hopf $*$-algebroid $\mathcal{A}(K(X), \widehat{\theta^{\mathrm{c}}}, \theta, \mu_\nu)$ with total algebra given by

$$\begin{array}{rccc} A' = & \mathbb{C}[G] \#_{\theta^{\mathrm{c}}} K(X) & \cong & \mathbb{C}[G \ltimes X] \\ & \lambda_g \# f & \mapsto & \displaystyle\sum_{x \in X} f(x)\lambda_{(g,x)} \end{array} .$$

Moreover

$$\begin{array}{rccc} \alpha_{\theta^{\mathrm{c}}}: & K(X) & \to & \mathbb{C}[G] \#_{\theta^{\mathrm{c}}} K(X) \\ & f & \mapsto & \displaystyle\sum_{x \in X} f(x)\lambda_{(e,x)} \end{array} , \qquad \begin{array}{rccc} \beta_{\widehat{\theta^{\mathrm{c}}}}: & K(X) & \to & \mathbb{C}[G] \#_{\theta^{\mathrm{c}}} K(X) \\ & f & \mapsto & \displaystyle\sum_{x \in X} f(x)\lambda_{(e,x)} \end{array} .$$

Algebraic quantum groups of compact/discrete type: Consider the trivial braided commutative measured Yetter-Drinfeld $\mathbb{G}^{\mathrm{c}}$-$*$-algebra $(\mathbb{C}, \theta = \mathsf{trv}, \widehat{\theta} = \mathsf{trv}, \mathrm{id}_{\mathbb{C}})$. In this case, we have $\gamma_\theta = \mathrm{id}$ and

$$\mathcal{A}(N, \theta, \widehat{\theta}, \mu) \cong \mathbb{G} \quad \text{and} \quad \mathcal{A}(N^{\mathrm{op}}_{\hat{\gamma}_\theta}, \widehat{\theta}^{\mathrm{c}}, \theta^{\mathrm{c}}, \mu^\circ) = \widehat{\mathbb{G}}^\circ.$$

Heisemberg algebras as algebraic quantum groupoids of discrete type: Let $\mathbb{G}$ be an algebraic quantum group of compact type. Then the total algebra of the Pontrjagin dual of the measured Hopf $*$-algebroid

$$\mathcal{A}(\mathcal{O}(\mathbb{G}), (S_{\mathbb{G}}^{-1} \otimes \mathrm{id}) \circ \Sigma \circ \Delta_{\mathbb{G}}, \mathrm{Ad}_{\Sigma(U^*)}, \varepsilon_{\mathbb{G}}),$$

is given by the opposite Heinseberg algebra

$$A' \cong (\mathcal{O}(\mathbb{G}) \#_{\blacktriangleright} \widehat{\mathcal{O}(\mathbb{G})})^{\mathrm{op}}_{S_{\mathbb{G}}^2 \# \widehat{S}_{\mathbb{G}}^{-2}}.$$

Here the canonical automorphisms are given by $S_{\mathbb{G}}^{-2}$ and $S_{\mathbb{G}}^2$.

C*-algebraic quantum transformation groupoids

Let $(N, \theta, \widehat{\theta}, \mu)$ be a braided commutative measured Yetter-Drinfeld $\mathbb{G}^{\mathrm{c}}$-$*$-algebra. Then:

Theorem (C*-LCQG[d]s arising from algebraic QTG[d]s, T. '18 + T. '23)

There is a Hopf C^*-bimodule over the base $C_r^*(N)$ with invariant C^*-valued weights denoted by $\mathfrak{G}_r(N, \theta, \widehat{\theta}, \mu)$. This object is the C^*-counterpart of the measured quantum transformation groupoid $\mathfrak{G}_{vN}(N, \theta, \widehat{\theta}, \mu)$. Moreover, using a C^*-pseudo-multiplicative unitary arising from $(N, \theta, \widehat{\theta}, \mu)$, we have a duality of Hopf C^*-bimodules over a base between the C^*-algebraic quantum transformation groupoids $\mathfrak{G}_r(N, \theta, \widehat{\theta}, \mu)$ and $\mathfrak{G}_r(N^{\mathrm{op}}_{\hat{\gamma}_\theta}, \widehat{\theta}^{\mathrm{c}}, \theta^{\mathrm{c}}, \mu^{\circ})$.

Examples:

- Compact/discrete transformation groupoids (*c.f.* Vallin, Timmermann)
- Quantum transformation groupoids arising from Fell bundles over discrete groups
- Compact/discrete quantum groups (trivial Yetter-Drinfeld algebras)
- Quantum transformation groupoids arising from quotient type coideals of compact quantum groups (*c.f.* Enock-Timmermann)
- Quantum transformation groupoids arising from quantum Bernoulli shift actions of discrete quantum groups (Ongoing work based on a Timmermann's idea)

Thanks for your attention